Acceleration of the Numerical Solution of Parabolic Singularly Perturbed Problems by Parallel Methods Based on a Defect-Correction Technique *

P.W. Hemker

CWI, Amsterdam, The Netherlands

G.I. Shishkin

Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Science, Ekaterinburg, Russia

L.P. Shishkina

Scientific Research Institute of Heavy Machine Building, Ekaterinburg, Russia

On the rectangular space-time domain $G = D \times (0, T]$ with the boundary $S = \overline{G} \setminus G$, where $D = (0, d)$, we consider the Dirichlet problem for the singularly perturbed parabolic equation:

$$\varepsilon^2 \frac{\partial}{\partial x} \left(a(x,t) \frac{\partial}{\partial x} u(x,t) \right) - c(x,t)u(x,t) - p(x,t) \frac{\partial}{\partial t} u(x,t) = f(x,t), \qquad (x,t) \in G, \tag{1a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S. \tag{1b}$$

In (1) $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions, moreover, $0 < a_0 \le a(x,t)$, $0 < p_0 \le p(x,t)$, $c(x,t) \ge 0$, $(x,t) \in \overline{G}$. The perturbation parameter ε may take any small positive value, say $\varepsilon \in (0,1]$. If $\varepsilon = 0$, the parabolic equation (1a) degenerates into a first-order equation, in which only the time derivative remains.

For small values of the parameter ε, as $\varepsilon \to 0$, the solution of (1) exhibits parabolic boundary layers (i.e., layers described by a parabolic equation) in some neighbourhood of the lateral boundary.

The presence of thin layers gives rise to difficulties when classical discretization methods are used for the numerical solution of problem (1). The error in the approximate solution depends on the value of ε and may be of a size comparable with

*This research was supported in part by the Netherlands Organization for Scientific Research NWO, dossiernr. 047.003.017, and by the Russian Foundation for Basic Research under Grant N 98-01-00362.

the exact solution. Therefore, an adapted placement of the mesh points is needed to insure that the error is independent of ε and depends only on the number of the nodal points used. Special schemes with this property are called *ε-uniformly convergent*. In this paper we study such ε-uniformly convergent schemes, which combine a standard finite difference operator (see, e.g. [1]) and a grid selection criterion for the space discretization (see, e.g., [2]).

Special ε-uniformly convergent difference schemes for singularly perturbed parabolic problems are fairly well developed in [2, 3, 4, 5] and in many other publications. If the data of problem (1) are sufficiently smooth, the order of ε-uniform convergence for the scheme that was studied is $\mathcal{O}(N^{-2}\ln^2 N + N_0^{-1})$, where $N+1$ and N_0+1 are, respectively, the number of nodes in the space and time meshes. For this scheme the amount of computational work is primarily determined by the time discretization, which is of first-order accuracy only, while the key difficulty for such a problem lies essentially in the space direction where we have second-order accuracy. Therefore, it is of interest to improve the accuracy in time, without essentially increasing the amount of computational work.

In [6, 7] we have developed an algorithm which can achieve a higher order of accuracy with respect to the time variable. To obtain the better approximation in time, we used auxiliary discrete problems on the same time-mesh to correct the difference approximations. The improvement of the accuracy in time, maintaining ε-uniform convergence, is achieved by correction of the right-hand side. The order of consistency for the new "corrected" scheme is higher than that for the base scheme. Then we can repeat such a defect correction in a similar way. This method with a sequential correction of the local truncation error allows us to achieve the same order of accuracy in both variables and, moreover, a higher order of accuracy with respect to the time variables. Based on this defect-correction procedure, we can construct ε-uniformly convergent difference schemes with an arbitrary high order of accuracy in time

$$\mathcal{O}(N^{-2}\ln^2 N + N_0^{-k}), \quad k \geq 2$$

provided that the solution is sufficiently smooth. To obtain ε-uniform convergence, we used a piecewise uniform mesh on $[0, d]$ with nodes that are condensed by a special way in the neighbourhood of the boundary layers [2, 3, 5].

To improve the implementation of the algorithm, it is desirable to have robust and effective numerical methods for solving discrete equations, i.e., finite difference schemes, arising from the approximation of the parabolic singularly perturbed problem (1). In [8] parallel computational methods were introduced on the basis of the Schwartz alternating method (see, e.g., [9]) that allowed us to accelerate the numerical solution of singularly perturbed Dirichlet problems for parabolic reaction-diffusion equations in two space dimensions. In that paper the special scheme has the order of ε-uniform convergence $\mathcal{O}(N^{-1}\ln N + N_0^{-1})$, however, the same technique permits to show that by this method we can obtain the second order (up to a logarithmic factor) with respect to the space variable.

We propose a parallel computational method for solving the system of discrete equations which arise in that case, when the defect-correction technique is used for the better approximation of problem (1). To construct the method, we use an algorithm based on a modified Schwarz technique that admits parallel computations on $P > 1$ solvers (processors). This construction follows [8] and [10]. By this way, we can achieve a higher order of accuracy in time, maintaining ε-uniform convergence and the second-order accuracy in space, as well as a high efficiency of the algorithms due to parallel computations. In this sense, the present algorithm is an improvement over a previously developed one. A further advantage is that the parallel defect-correction procedure can be easily implemented. We show that the parallel method does not require iterations at each time level. Thus, the practical efficiency of acceleration for this method is proportional to the number of parallel solvers used.

According to the classical Schwartz method [9], the original domain D is decomposed into several overlapping subdomains D^k, $k = 1, \ldots, K$:

$$D = \bigcup_{k=1}^{K} D^k. \tag{2}$$

Assume $G^k = D^k \times (0, T]$, $k = 1, \ldots, K$. The conventional Schwartz method for analytically solving boundary value problems depends on the subsets D^k, which form the partition (2) of the original domain D, on the width of the subdomain overlapping as well as on the initial function in the iterative process. The Schwartz method, which admits parallel computations on $P > 1$ processors, also depends on the number P of parallel solvers. The difference schemes based on the Schwartz alternating method depends, in addition, on the distribution of the mesh nodes in the subdomains D^k.

Because of a sufficiently large of the parameters that determine the alternating difference schemes based on the Schwartz method, it may turn out that these schemes do not converge ε-uniformly or using parallel domain decomposition schemes do not accelerate the numerical solution of problem (1). It is shown that a modified Schwartz discrete method, which uses a standard finite difference operator on piecewise uniform meshes and a defect-correction procedure similar to [7], yields approximations that converge, as N, $N_0 \to \infty$, to the solution of problem (1) ε-uniformly with the rate of convergence $\mathcal{O}(N^{-2} \ln^2 N + N_0^{-k})$, $k = 1, 2, 3$, if the following condition is fullfilled:

$$\rho = \rho(\varepsilon) > 0, \quad \varepsilon \in (0, 1], \quad \inf_{\varepsilon \in (0,1]} [\, \varepsilon^{-1} \rho(\varepsilon) \,] > 0. \tag{3}$$

By ρ we denote the minimal width of the overlapping of the subsets D^k, $k = 1, \ldots, K$. The value of ρ may generally depend on ε, say $\rho = \rho(\varepsilon)$. Note that the condition (3) is necessary and sufficient for ε-uniform convergence (in the maximum norm).

The principal question here is: does the use of parallel computations really accelerate the solution of the difference schemes? Answering this question, we give a few assumptions and the conditions under which an increasing number P of solvers in parallel computations leads to the acceleration of the process of the numerical solution uniformly in the perturbation parameter ε. We show also that the extreme error of the alternating difference scheme (for the worst initial function in the iterative process) does not increase.

Thus, using the parallel domain decomposition method on the basis of the Schwartz technique, we have constructed the parallel defect correction procedure and new iterative schemes that can be easily implemented in order to improve the accuracy in time, still obtaining the second-order ε-uniform accuracy in the space discretization, as well as to accelerate the process of the numerical solution uniformly in ε due to parallel computations.

References

[1] A.A. Samarsky. Theory of Difference Scheme. Nauka, Moscow, 1989 (in Russian, 3rd edition); German transl., Akademische Verlag, Geest & Portig, Leipzig, 1984.

[2] J.J.H. Miller, E. O'Riordan and G.I. Shishkin. Fitted Numerical Methods for Singular Perturbation Problems. World Scientific, Singapore, 1996.

[3] P.A. Farrell, P.W. Hemker, and G.I. Shishkin. Discrete approximations for singularly perturbed boundary value problems with parabolic layers. I, *J. Comput. Math.*, 1996, **14** (1), 71–97. II, *J. Comput. Math.*, 1996, **14** (2), 183–194. II, *J. Comput. Math.*, 1996, **14** (3), 273–290.

[4] H.-G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations. Springer-Verlag, Berlin, 1996.

[5] G.I. Shishkin. Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Ural Branch of Russian Acad. Sci., Ekaterinburg, 1992 (in Russian).

[6] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. The use of defect correction for the solution of parabolic singular perturbation problems. *ZAMM,* 1997, bf 77 (1), 59–74.

[7] P.W. Hemker, G.I. Shishkin and L.P. Shishkina. $\mathcal{E}$-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems. *IMA J. Numer. Anal.*, 1998, **14** (4), 1–24.

[8] G.I. Shishkin. Acceleration of the process of the numerical solution to singularly perturbed boundary value problems for parabolic equations on the basis of parallel computations. *Russ. J. Numer. Anal. Math. Modelling*, 1997, **12** (3), 271–291.

[9] G.I. Marchuk. Methods of Numerical Mathematics. Springer, New York, 1982.

[10] G.I. Shishkin, P.N. Vabishchevich. Parallel domain decomposition methods with the overlapping of subdomains for parabolic problems. *Math. Models and Methods in Applied Sciences*, 1996, **6** (8), 1169–1185.