PARALLEL METHODS FOR QUASILINEAR SINGULARLY PERTURBED REACTION-DIFFUSION EQUATIONS [1]

P.W. Hemker, G.I. Shishkin, I.V. Tselishcheva
(CWI, Amsterdam, the Netherlands; Ekaterinburg, Russia,
Institute of Mathematics and Mechanics UB RAS)

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On the rectangle $\overline{D}$, where $D = \{x : \ 0 < x_s < d_s, \ s = 1,2\}$ we consider Dirichlet's problem for the quasilinear singularly perturbed elliptic equation

$$L_{(1)}(u(x)) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} u(x) - g(x, u(x)) = 0, \qquad x \in D, \tag{1a}$$

$$u(x) = \varphi(x), \qquad x \in \Gamma. \tag{1b}$$

Here $\Gamma = \overline{D} \setminus D$, the functions $a_s(x)$, $g(x,u)$ and $\varphi(x)$ are sufficiently smooth, respectively, on the sets $\overline{D}$, $\overline{D} \times R$ and on the sides Γ_j, $\Gamma = \cup_j \Gamma_j$, $j = 1, ..., 4$, $\varphi(x)$ is continuous on Γ. Assume $0 < a_0 \le a_s(x) \le a^0$, $x \in \overline{D}$,

$$(\partial / \partial u)\, g(x,u) \ge g_0 > 0 \quad \text{for all} \quad (x,u) \in \overline{D} \times \left[-M_{(2)}, M_{(2)} \right], \tag{2}$$

where $M_{(2)}$ is a sufficiently large number. The perturbation parameter ε takes arbitrary values from the half-interval $(0,1]$; say $\varepsilon \ll 1$.

Model problems of similar kind arise, for example, in numerical modelling of stationary diffusion processes accompanied by first-order chemical reactions. The parameter ε characterizes the diffusion coefficient of the involved matter, and the constant g_0 refer to the reaction rate.

As $\varepsilon \to 0$, regular boundary layers appear in small neighbourhoods of the smooth parts of the boundary Γ, and corner (elliptic) layers appear in a neighbourhood of the set Γ_0 of the corner points. The data in (1) are assumed to satisfy the necessary compatibility conditions on Γ_0.

For problem (1) we construct an iterative domain decomposition scheme, that converges ε-uniformly and also allows for parallel computations.

Let us first give an iteration-free difference scheme. On the set $\overline{D}$ we introduce the rectangular grid

$$\overline{D}_h = \overline{\omega}_1 \times \overline{\omega}_2, \tag{3}$$

where $\overline{\omega}_s = \left\{ x_s^i : \ 0 = x_s^0 < ... < x_s^{N_s} = d_s \right\}$ is a (possibly) nonuniform mesh

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on $[0, d_s]$, $s = 1, 2$. Define $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i, x_s^{i+1} \in \overline{\omega}_s$; let $h \le MN^{-1}$, where[2] $h = \max_{i,s} h_s^i$, $N = \min[N_1, N_2]$.

We approximate problem (1) by the finite difference scheme

$$\Lambda_{(4)}(z(x)) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x) \delta_{\overline{xs}\,\widehat{xs}}\, z(x) - g(x, z(x)) = 0, \quad x \in D_h, \tag{4a}$$

$$z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{4b}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\delta_{\overline{xs}\,\widehat{xs}} z(x) = z_{\overline{xs}\,\widehat{xs}}(x)$ is the second-order (central) difference derivative on the non-uniform mesh $\overline{\omega}_s$ [1].

The difference scheme (4), (3) is monotone [1]. Using majorant functions and taking into account certain *a priori* estimates (see, e.g., [2]), we find: $|\, u(x) - z(x)\,| \le M\varepsilon^{-1}N^{-1}$, $x \in \overline{D}_h$. So this scheme converges, as $N \to \infty$, for fixed values of the parameter ε, but it does not converge ε-uniformly.

Note that we evaluate the errors in the maximum norm (in L_∞). Other norms, e.g., the L_p-norms $(1 \le p < \infty)$ and the weighted energy norm $||\cdot||_\varepsilon$, are unsatisfactory: in these norms the boundary layer vanishes for $\varepsilon \to 0$.

Now we introduce the special grid depending on ε

$$\overline{D}_h^\star = \overline{\omega}_1^\star \times \overline{\omega}_2^\star. \tag{5}$$

Here $\overline{\omega}_s^\star = \overline{\omega}_s^\star(\sigma_s)$ is a piecewise uniform mesh refined in the neighbourhood of the end-points of $[0, d_s]$. In each interval $[0, \sigma_s]$ and $[d_s - \sigma_s, d_s]$ we use a fine mesh with step-size $h_s^{(1)} = 4\sigma_s N_s^{-1}$, and in $[\sigma_s, d_s - \sigma_s]$ we use a coarse mesh with step-size $h_s^{(2)} = 2(d_s - 2\sigma_s)N_s^{-1}$. We take $\sigma_s = \min[4^{-1}d_s,\ m_1^{-1}\,\varepsilon \ln N_s]$, where $0 < m_1 < m_0$, $m_0 = (g_0/a^0)^{1/2}$.

Let the solution $U_0(x)$ of the reduced equation $g\,(x, U_0(x)) = 0$, $x \in \overline{D}$, satisfy the estimate

$$\left|\, \partial^k U_0(x) \,/\, \partial x_1^{k_1}\, \partial x_2^{k_2} \,\right| \le M_{(6)}, \quad x \in \overline{D}, \quad k \le 2. \tag{6}$$

Theorem 1. *Assume in* (1) *that* $a_1, a_2 \in C^l(\overline{D})$, $g \in C^{l,l}(\overline{D} \times [-M_1, M_1])$, $\varphi \in C^l(\overline{D})$, *and let* $u \in C^l(\overline{D})$, $l = 4 + \alpha$, $\alpha > 0$, *where* $M_1 = M_0 M_{(6)}$, $M_0 > 18$, *and also let condition* (2) *be true with* $M_{(2)} = M_1$. *Then the difference scheme* (4), (5) *converges, as* $N \to \infty$, ε*-uniformly with an error bound given by*

$$|\, u(x) - z(x)\,| \le MN^{-2} \ln^2 N, \quad x \in \overline{D}_h^\star.$$

Let the connected sets

$$D^k, \quad k = 1, \dots, K \tag{7}$$

with piecewise smooth boundaries Γ^k, $\Gamma^k = \overline{D}^k \setminus D^k$, cover the domain D:

[2] Here and below M, M_i (m, m_i) denote sufficiently large (small) positive constants independent of ε.

$D = \bigcup_{k=1}^{k} D^k$. We denote by $D^{[k]}$ the union of the subdomains $D^1, \dots, D^K$ excluding the set D^k: $D^{[k]} = \bigcup_{i=1,\ i \neq k}^{K} D^i$. We denote the minimal overlap of the sets D^k and $D^{[k]}$ by Δ^k, and also let Δ denote the least value of Δ^k, $k = 1, \dots, K$, i.e.

$$\Delta = \min_{k,\, x^1,\, x^2} \rho(x^1, x^2), \quad x^1 \in \overline{D}^k, \ x^2 \in \overline{D}^{[k]}, \quad x^1, x^2 \notin \left\{ D^k \cap D^{[k]} \right\}, \tag{8}$$

$k = 1, \dots, K$, where $\rho(x^1, x^2)$ is the distance between the points $x^1, x^2 \in \overline{D}$. In general, the value Δ may depend on the parameter ε: $\Delta = \Delta(\varepsilon)$.

Now we construct a Schwartz method modified for implementation in a parallel environment with $P \geq 1$ processors. This construction follows [3,4].

Let each subdomain D^k, $k = 1, \dots, K$ from (7) be partitioned in P disjoint non-overlapping parts (some of them may be empty): $D^k = \bigcup_{p=1}^{P} D_p^k$, $k = 1, \dots, K$, $\overline{D}_i^k \cap \overline{D}_j^k = \emptyset$, $i \neq j$. Assume $\Gamma_p^k = \overline{D}_p^k \setminus D_p^k$.

On the sets $\overline{D}_p^k$ we construct the coherent meshes

$$\overline{D}_{p\,h}^k = \overline{D}_p^k \cap \overline{D}_h, \qquad k = 1, \dots, K, \quad p = 1, \dots, P, \tag{9a}$$

where $\overline{D}_h = \overline{D}_{h(3)}$ or $\overline{D}_h = \overline{D}_{h(5)}^\star$. Let $\Gamma_{p\,h}^k = \overline{D}_{p\,h}^k \setminus D_p^k$.

Given the function $z^0(x)$ on $\overline{D}_h$ satisfying condition (4b), we find the sequence of the functions $z^r(x)$, $x \in \overline{D}_h$, $r = 1, 2, \dots$ from the solutions of the discrete problems

$$\left.\begin{array}{ll} \Lambda_{(4)}\left(z_p^{r+\frac{k}{K}}(x)\right) = 0, & x \in D_{p\,h}^k, \\ z_p^{r+\frac{k}{K}}(x) = z^{r+\frac{k-1}{K}}(x), & x \in \Gamma_{p\,h}^k \end{array}\right\}, \quad p = 1, ..., P; \tag{9b}$$

$$z^{r+\frac{k}{K}}(x) = \left\{ \begin{array}{ll} z_p^{r+\frac{k}{K}}(x), & x \in \overline{D}_{p\,h}^k, \quad p = 1, ..., P, \\ z^{r+\frac{k-1}{K}}(x), & x \in \overline{D}_h \setminus \overline{D}^k \end{array} \right\}, \quad \begin{array}{l} x \in \overline{D}_h, \\ k = 1, \dots, K, \end{array}$$

$$z^{r+1}(x) = z^{r+\frac{K}{K}}(x), \qquad x \in \overline{D}_h; \qquad r = 0, 1, 2, \dots \ . \tag{9c}$$

Under the condition

$$\Delta = \Delta_{(8)}(\varepsilon) > 0, \quad \varepsilon \in (0, 1], \qquad \inf_{\varepsilon \in (0,1]} \left[\varepsilon^{-1} \Delta_{(8)}(\varepsilon) \right] > 0, \tag{10}$$

we obtain the estimate (see [3] for the linear case)

$$|\, z(x) - z^r(x) \,| \leq M q^r, \quad x \in \overline{D}_h, \tag{11}$$

where $q \leq 1 - m$, e.g., we can take $q = \exp\bigl(-m_0 \varepsilon^{-1} \Delta\bigr)$. If condition (10) is violated, the functions $z^r(x)$ do not converge ε-uniformly with respect to the number r of iterations. On the special grid $\overline{D}^{\star}_{h(5)}$ we have the estimate

$$|\, u(x) - z^r(x) \,| \leq M \left[N^{-2} \ln^2 N + q^r \right], \quad x \in \overline{D}^{\star}_h. \tag{12}$$

Emphasize that the estimate for q in (11), (12) is independent of ε.

Theorem 2. *The condition* (10) *is necessary and sufficient for ε-uniform convergence (as $r \to \infty$) of the functions $z^r(x)$, i.e., the solutions of the decomposition scheme* (9), (3) *with P parallel solvers, to the solution $z(x)$ of the base scheme* (4), (3). *If the hypotheses of Theorem* 1 *and also condition* (10) *are fulfilled, the solutions $z^r(x)$ of scheme* (9), (5) *converge, as $N, r \to \infty$, to the solution $u(x)$ of the boundary value problem* (1) *ε-uniformly. Under condition* (10), *the estimates* (11), (12) *are valid.*

The difference scheme is nonlinear. To solve the problem, we apply a linearization procedure [1], replacing $g(x, z^r(x))$ by $c(x)(z^r_{(i)}(x) - z^r_{(i-1)}(x)) + g(x, z^r_{(i-1)}(x))$. Here $z^r_{(i)}(x)$ is the i-th inner iteration, the function $c(x)$ satisfies the condition $c(x)u - g(x,u) \geq c_0 u$, $c_0 > 0$, where $(\partial/\partial u) g(x,u) \leq g^0$, $(x,u) \in \overline{D} \times [\,-M_{(2)}, M_{(2)}\,]$.

Thus, for parallelization of the computational method, we have constructed iterative difference schemes that converge ε-uniformly with respect to both the number N of grid nodes and the number r of (outer) iterations required for convergence of the iterative process.

References

[1] Samarsky A.A. Theory of Difference Schemes. Nauka, Moscow, 1989.

[2] G.I. Shishkin. Grid Approximations of Singularly Perturbed Elliptic and Parabolic Equations. UB RAS, Ekaterinburg, 1992.

[3] Shishkin G.I., Tselishcheva I.V. Parallel methods of solving singularly perturbed boundary value problems for elliptic equations. Mat. Model. 1996. V. 8, N 3. P. 111–127 (in Russian).

[4] Shishkin G.I. Acceleration of the process of the numerical solution to singularly perturbed boundary value problems for parabolic equations on the basis of parallel computations. Russ. J. Numer. Anal. Math. Modelling. 1997. V. 12, N 3. P. 271–291.

e-mail: Grigorii@shishkin.ural.ru, tsi@shishkin.ural.ru